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Direct contact heat transfer in spherical geometry associated with phase transformation—a closed-form solution

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NOMENCLATURE

- Bi Biot number, hR_0/k
- C specific heat capacity
- $\bar{F}$ dimensionless average temperature of the sphere, $(T_m - \bar{T})/(T_m - T_0)$
- h liquid–film heat transfer coefficient
- h_{sf} latent heat of melting of the metal
- k thermal conductivity
- r radial coordinate
- R time-dependent radius of the sphere
- R_0 radius of the sphere at time zero
- R^+ dimensionless variable radius of the sphere, R/R_0
- S subcooling parameter of the sphere, $C(T_m - T_0)/h_{sf}$
- T temperature
- $\bar{T}$ average temperature of the sphere.

Greek symbols

- α thermal diffusivity
- ρ density
- τ time
- τ^+ dimensionless time, $\alpha\tau/R_0^2$
- φ degree of superheat parameter, $(T_l - T_m)/(T_m - T_0)$.

Subscripts

- 0 at time zero
- l liquid
- m melting.

INTRODUCTION

THE PHENOMENON of transient heat conduction taking place in spherical objects associated with phase transformation is encountered in various processes of industrial importance such as melting of metal pellets in a molten metal bath in metallurgical operations, condensation of saturated vapor on liquid droplets in jet condensers, evaporation of droplets in superheated vapor medium in desuperheaters and fluidized-bed coating of a polymeric resin on spherical metal objects in processes of protective coating, etc. According to El-Genk and Cronenberg [1] problems of one-dimensional transient heat conduction in plane geometry involving change of phase can be classified into three categories, viz. problems of (i) continuous growth, (ii) asymptotic growth and (iii) growth and decay of the phase-change front. However, the spherical geometry rules out the possibility of continuous growth due to the adiabatic condition of heat flow at the center of the sphere. In literature, the problems involving phase transformation in spherical geometry are numerically solved employing different mathematical techniques [2, 3]. Ehrich *et al.* [2] employed an integral method with the aid of a Green's function to obtain the growth and decay of a metal sphere immersed in its own melt. Brounshtein *et al.* [3] solved the problem of condensation of a saturated vapor on a spherical liquid droplet using a nonlinear variational differences scheme to predict the rate of condensation of vapor on the liquid drop during its asymptotic growth. However, in view of the technical importance of these problems an explicit expression is obtained for the asymptotic growth and the growth-and-decay in change-of-phase problems in spherical geometry. The closed-form solution obtained in the present analysis is in good agreement with the numerical results [2, 3] and experimental data [4] available in literature.

PHYSICAL MODEL

The model considered is the freezing/melting problem on the surface of a metal sphere of initial radius R_0 and at a temperature T_0 immersed in its own melt, which is maintained at a constant bulk temperature T_l, where $T_l > T_m$. For $\tau > 0$, in the initial stages a frozen shell forms on the sphere, the rate of growth depends on the rate of thermal accumulation in the solid sphere. However, subsequently the convective heat flow from the liquid metal bath causes melting of the frozen shell and the immersed metal sphere. Thus the problem falls under the third category involving growth-and-decay of the metal sphere according to the classification of El-Genk and Cronenberg [1]. Further, it is relevant to point out that the present formulation also holds good for the physical problems, viz. condensation of a saturated vapor on a spherical liquid droplet (a case of asymptotic growth of the droplet) and evaporation of a droplet in a quiescent medium of a superheated vapor (a case of growth-and-decay of the droplet).

ANALYSIS

For a spherical metal pellet or for a liquid droplet with low Péclet numbers which preclude internal circulation, the heat flow into the sphere is governed by the following unsteady-state heat conduction equation in the sphere:

$$\rho C \frac{\partial T}{\partial \tau} = \frac{k}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \quad \text{for} \quad 0 \leqslant r \leqslant R. \tag{1}$$

The boundary conditions are as follows:

$$r = 0, \quad \frac{\partial T}{\partial r} = 0 \tag{2}$$

$$r = R, \quad T = T_m. \tag{3}$$

The heat balance at the moving change-of-front (at $r = R$) gives:

$$k \left.\frac{\partial T}{\partial r}\right|_{r=R} = \rho h_{sf} \frac{dR}{d\tau} + h(T_l - T_m). \tag{4}$$

The problem is tackled by employing an integral method of approach. Thus, integration of (1) with respect to r between the limits 0 and R yields the following equation while utilizing the boundary condition, equation (3)

$$\rho C \frac{d}{d\tau} \int_0^R r^2 (T - T_m)\, dr = kR^2 \left. \frac{\partial T}{\partial r} \right|_{r=R}. \quad (5)$$

Equation (5) with the aid of the equation of heat balance, i.e. equation (4) gives

$$\frac{\rho C}{R^2} \frac{d}{d\tau} \int_0^R r^2 (T - T_m)\, dr = \rho h_{sf} \frac{dR}{d\tau} + h(T_l - T_m). \quad (6)$$

The following temperature profile in the sphere, i.e. for $0 \leqslant r \leqslant R(\tau)$ is chosen:

$$\frac{T_m - T}{T_m - T_0} = -\frac{2}{\pi} \frac{R}{r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi \frac{r}{R} \exp(-n^2 \pi^2 \alpha \tau / R^2). \quad (7)$$

Equation (7) satisfies the boundary conditions, (2) and (3). Evaluation of equation (6) with the aid of the temperature profile (7) yields the following differential equation in dimensionless notation:

$$\frac{dR^+}{d\tau^+} = (AR^+ - SBi\varphi)/(1 + 2\tau^+ A + B) \quad (8)$$

where

$$A = \frac{2S}{R^{+2}} \sum_{n=1}^{\infty} \exp(-n^2 \pi^2 \tau^+ / R^{+2})$$

$$B = \frac{6S}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-n^2 \pi^2 \tau^+ / R^{+2}).$$

Equation (8) is numerically solved by a fourth-order Runge–Kutta method with the initial condition that at $\tau^+ = 0$, $R^+ = 1$. Equation (8) yields solutions for two different situations, namely the case of growth-and-decay of the sphere (for $\varphi > 0$) and the case of asymptotic growth of the sphere (for $\varphi = 0$). It is found that the numerical results of (8) agree well with the earlier analytical [2, 3] and experimental [4] results as shown in Figs. 1–3 validating the present analysis for the range of parameters investigated. The agreement further prompted us to make an attempt to obtain an approximate closed-form solution making use of the temperature profile (7) and incorporating the lumped system approximation as depicted below.

An average equation of heat balance between the sphere and the bulk medium that surrounds the sphere is written as shown below:

$$\rho C \frac{4}{3} \pi R^3 \frac{d\bar{T}}{d\tau} = \rho 4\pi R^2 [C(T_m - \bar{T}) + h_{sf}] \times \frac{dR}{d\tau} + 4\pi R^2 h(T_l - T_m) \quad (9)$$

where $\bar{T}$ is the average temperature of the sphere defined by the following equation:

$$\bar{T} = \frac{3}{R^3} \int_0^R r^2 T\, dr. \quad (10)$$

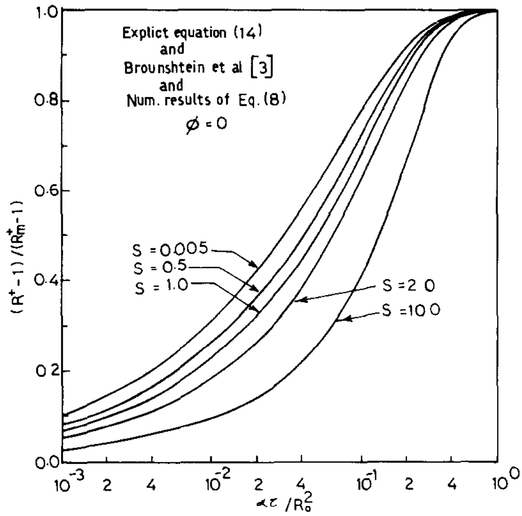

FIG. 2. Comparison of the closed-form solution with the numerical results of Brounshtein *et al.* [3], case of $\varphi = 0$.

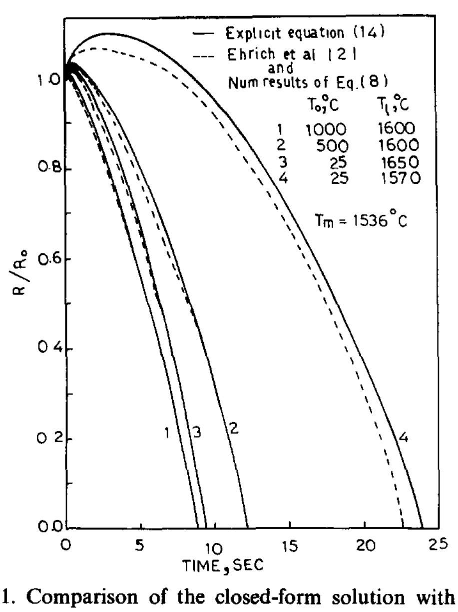

FIG. 1. Comparison of the closed-form solution with the numerical results of Ehrich *et al.* [2], effect of T_0 and T_l.

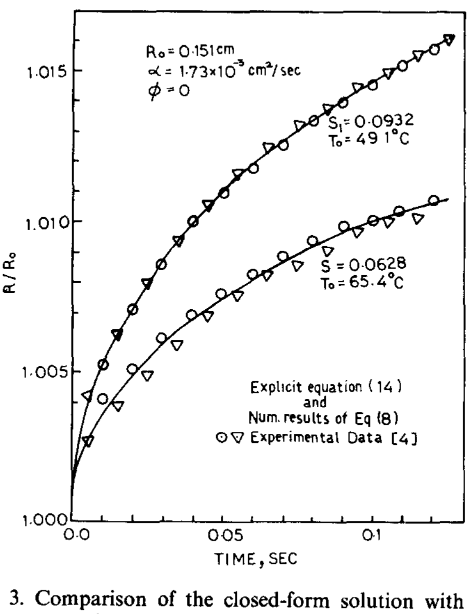

FIG. 3. Comparison of the closed-form solution with the experimental data of Ford and Lekic [4].

Equation (9) is valid in the case of a metal sphere possessing high thermal conductivity or in the case of a liquid droplet having very small radius. Equation (9) is written in dimensionless form as follows:

$$\frac{SR^+}{3}\frac{d\bar{F}}{d\tau^+}+(1+S\bar{F})\frac{dR^+}{d\tau^+} = -SBi\varphi. \quad (11)$$

There are two unknown variables in (11), namely $\bar{F}$ and R^+. It is assumed that $\bar{F}$ is explicitly related to τ^+, because of the lumped parameter approximation, as given by

$$\bar{F} = \exp(-\beta\tau^+) \quad (12)$$

where β is a coefficient whose value is to be indirectly determined.

Equation (12) satisfies the conditions that

$$\tau^+ = 0,\ \bar{F} = 1 \text{ and } \tau^+ \to \infty,\ \bar{F} \to 0. \quad (13)$$

Solution of (11) with the help of (12) and with the initial condition that at $\tau^+ = 0$, $R^+ = 1$ gives the following expression for R^+:

$$R^+ = [R_m^+ - SBi\varphi(G_1-G_2)/2\beta]/x \quad (14)$$

where

$$R_m^+ = (S+1)^{1/3}$$

$$x = [1+S\exp(-\beta\tau^+)]^{1/3}$$

$$G_1 = 3\ln\left(\frac{R_m^+-1}{x-1}\right)-\beta\tau^+$$

$$G_2 = 2\sqrt{3}\left[\arctan\left(\frac{2R_m^++1}{\sqrt{3}}\right)-\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right].$$

An expression for the coefficient β is obtained with the aid of equations (7), (10) and (12) and utilizing the numerical results of the integral method of analysis as follows.

Equation (10) with the help of the temperature profile (7) gives

$$\bar{F} = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2\tau^+/R^{+2}). \quad (15)$$

With the help of the numerical results of (8) and (15) for the range of parameters shown in Figs. 1–3, the following expression for the coefficient β is chosen:

$$\beta = \frac{1}{\tau^+}\ln\left(\frac{1}{Q}\right) \quad (16)$$

where

$$Q = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2\tau^+).$$

Thus (14) with the aid of (16) provides an explicit expression for R^+ as a function of τ^+ and the system parameters, S, Bi, and φ. It would be evident that (14) agrees well with the numerical results for the range of parameters investigated.

In other words an implied feature in the derivation of equation (14) is that a simple iterative procedure is adopted first treating β as time invariant and subsequently correcting β for better accuracy with the real physical process.

COMPARISON OF RESULTS

A comparison of the present closed-form solution given by equation (14), with the aid of (16), with the numerical results of Ehrich *et al.* [2] as indicated in Fig. 1 shows good agreement with the maximum deviation of $\pm 5\%$ between the two. In Fig. 1 the effects of T_0 and T_i on the growth-and-decay of the sphere are shown taking the properties of sponge iron [2]. In other words, Fig. 1 shows the effect of S (the subcooling parameter) and φ the degree-of-superheat parameter. It can be observed that with an increase in the degree of subcooling there is an increase in the growth of the solidified shell. The effect of the superheat of the liquid over its solidification temperature is to impede the growth of the frozen crust and to accelerate the melting of the metal sphere. Further, in Fig. 2 equation (14) is compared with the numerical results of Brounshtein *et al.* [3] for the case of condensation of saturated vapor on a liquid droplet. It is evident from Fig. 2 that (14) is able to predict the condensation rates with reasonably good accuracy over a wide range of subcooling parameters. Also, equation (14) shows good agreement with the experimental data of Ford and Lekic [4] for condensation of saturated steam taking place on a water droplet as shown in Fig. 3. The agreement of (14) with the analytical and experimental results mentioned above over a wide range of system parameters, viz. for $0.005 \leq S \leq 10$ and $0 \leq Bi \leq 117$, suggests that (14) can predict the evaporation rates of a droplet in a quiescent medium of superheated vapor.

CONCLUSIONS

An approximate closed-form solution given by equation (14) is presented for ready use to a design engineer for the following three situations:

(1) Melting of a spherical metal pellet immersed in a liquid metal.
(2) Condensation of saturated vapor on a spherical liquid droplet.
(3) Evaporation of a spherical liquid droplet in a quiescent medium of a superheated vapor.

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